multi-vector physics

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In this book, Maxwell's equations are derived using multi-vectors and $curl$ instead of the well-known electromagnetic tensor. $curl$ is very similar to the exterior derivative of a differential form. In this process, we learn that vector potential, electric field, magnetic field, charge, current density, etc. are multi-vectors with different dimensions.

Furthermore, by extending the dimension of vector potential(field) using the same method, we can derive extended Maxwell's equations for the electromagnetic, weak, strong, and gravitational forces. Furthermore, we can understand the dimensions and properties of mass, charge, spin, color, and the Higgs.

Force can be directly derived from the energy density of the electromagnetic field. The electromagnetic force and the time-dependent force, i.e., Joule heating, are all obtained simultaneously. Furthermore, we learn why the weak force is called the weak force. Gravity and the strong force are also derived.

Surprisingly, in four dimensions, multi-vectors(basis) are sedenion(16 dimensional numbers). This is not difficult. After reading it, it will become a tool for interpreting physical quantities.

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1 multi-vector

In this book, vector means multi-vector. A multi-vector is obtained by production several existing 1-vectors using the convenient symbol wedge(∧). If $v_1, v_2, \dots, v_r$ are existing 1-vectors, $(v_1 \wedge v_2)$ is a 2-vector, and $(v_1 \wedge v_2 \wedge v_3)$ is a 3-vector.

A. anti-symmetry

A vector changes sign when the order of the wedge product is changed. It is anti-commutative.

$$(v_1 \wedge v_2) = -(v_2 \wedge v_1), \qquad (v_1 \wedge v_1) = 0$$

$$(v_1 \wedge v_2 \wedge v_3) = \varepsilon_{ijk}(v_i \wedge v_j \wedge v_k), \qquad i,j,k = 1,2,3, \qquad \varepsilon_{ijk}: Levi\text{-}Civita\ symbol$$

…

B. dot product

The dot product of vectors is commutative. If a, b, c are 1-vectors,

$$a \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r)$$
$$= (-1)^{i-1}(a \cdot v_i)(v_1 \wedge \dots \wedge \check{v}_i \wedge \dots \wedge v_r), \qquad \check{v}_i: v_i\ skip, \qquad Einstein\ summation$$
$$= (v_1 \wedge v_2 \wedge \dots \wedge v_r) \cdot a$$

ex) $a \cdot (v_1 \wedge v_2 \wedge v_3)$

$$= (a \cdot v_1)(v_2 \wedge v_3) - (a \cdot v_2)(v_1 \wedge v_3) + (a \cdot v_3)(v_1 \wedge v_2)$$

$$(a \wedge b) \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r) = b \cdot \{a \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r)\}$$

ex) $(a \wedge b) \cdot (v_1 \wedge v_2 \wedge v_3) = b \cdot \{a \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r)\}$

$$= b \cdot \{(a \cdot v_1)(v_2 \wedge v_3) - (a \cdot v_2)(v_1 \wedge v_3) + (a \cdot v_3)(v_1 \wedge v_2)\}$$
$$= \{(a \cdot v_1)(b \cdot v_2) - (a \cdot v_2)(b \cdot v_1)\}v_3 - \{(a \cdot v_1)(b \cdot v_3) - (a \cdot v_3)(b \cdot v_1)\}v_2$$
$$+\{(a \cdot v_2)(b \cdot v_3) - (a \cdot v_3)(b \cdot v_2)\}v_1$$

If we swap a and b in the equation above, or if $a = b$, we can also find the following properties.

$$(a \wedge b) \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r) = -(b \wedge a) \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r)$$

$$b \cdot \{a \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r)\} = -a \cdot \{b \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r)\}$$

$$(a \wedge a) \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r) = a \cdot \{a \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r)\} = 0$$

$$(a \wedge b \wedge c) \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r) = c \cdot [b \cdot \{a \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r)\}]$$

$$= c \cdot \{(a \wedge b) \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r)\} = (b \wedge c) \cdot \{a \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r)\}$$

If $w_1, w_2, \dots, w_r$ are also 1-vectors,

$$(w_1 \wedge w_2 \wedge \dots \wedge w_r) \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r) = \left|\left(v_i \cdot w_j\right)\right| = det\left(v_i \cdot w_j\right)$$

C. dual vector

In an n-dimensional space, an n-vector consisting of Cartesian coordinates basis $e_1, e_2, \dots, e_n$ is called a unit pseudoscalar I.

$$I = e_1 \wedge e_2 \dots \wedge e_n$$

If $v = (v_1 \wedge v_2 \wedge \dots \wedge v_r)$, then the dual vector of v, v^d is defined as follows.

$$v^d = v \cdot I \qquad \text{in particular} \quad I^d = 1, \;\; 1^d = I.$$

v and v^d are perpendicular to each other and are 1:1.

$$v \cdot v^d = (v_1 \wedge v_2 \wedge \dots \wedge v_r) \cdot \{(v_1 \wedge v_2 \wedge \dots \wedge v_r) \cdot I\}$$

$$=(v_1 \wedge v_2 \wedge \dots \wedge v_r \wedge v_1 \wedge v_2 \wedge \dots \wedge v_r) \cdot I = 0$$

For example, in 3D space

$$I = e_1 \wedge e_2 \wedge e_3, \qquad e_1^d = e_1 \cdot I = e_1 \cdot (e_1 \wedge e_2 \wedge e_3) = e_2 \wedge e_3$$

If we consider $e_2 \wedge e_3$ as a plane that physically consists of e_2 and e_3, then e_1 is a vector perpendicular to that plane. Duality is a term used in projective geometry, and pseudoscalar is a term used in geometric algebra.

2 curvilinear coordinates system

It is an arbitrary n-dimensional differentiable spatial coordinate system. The coordinates of a point are $P(p^1, p^2, \dots, p^n)$, which means the vector from the origin O to P.

A. basis

At point $P(p^1, p^2, \dots, p^n)$, the basis u_i is defined as follows.

$$u_i = \frac{\partial P}{\partial p^i}$$

B. reciprocal basis

When $u^i \cdot u_j = \delta^i_j$, $u^1, u^2 \dots u^n$ are reciprocal basis at P. Assuming matrices g^{ij} and g_{jk}, u^i at that point can be expressed as u_j, and u_i can be expressed as u^j.

$$u^i = g^{ij}u_j, \qquad u^i \cdot u^k = g^{ij}u_j \cdot u^k = g^{ik}$$

$$u_i = g_{ij}u^j, \qquad u_i \cdot u_k = g_{ij}u^j \cdot u_k = g_{ik}$$

Here, g_{jk} and g^{ij} are inverse matrix relationships.

$$u^i \cdot u_k = g^{ij}u_j \cdot u_k = g^{ij}g_{jk} = \delta^i_k$$

Therefore, by finding the basis at P, finding the matrix $g_{ij} = u_i \cdot u_j$, and finding the inverse matrix g^{ij}, we can find u^i of that point.

C. infinitesimal volume $[dV]_r$

$[dV]_r$ is an r-vector. It means something like a very small r-dimensional parallelepiped. The reference vertex is $P(p^1, p^2, \dots, p^n)$.

$$[dV]_r = \left(u_{\sigma_1} \wedge u_{\sigma_2} \wedge \dots \wedge u_{\sigma_r}\right) dp^{\sigma_1} dp^{\sigma_2} \dots dp^{\sigma_r}$$

$$\{\sigma_1, \sigma_2, \dots, \sigma_r\} \subset \{1,2, \dots, n\}, \qquad (\sigma_1 < \sigma_2 < \dots < \sigma_r)$$

The r edges are $u_{\sigma_1} dp^{\sigma_1}, u_{\sigma_2} dp^{\sigma_2}, \dots, u_{\sigma_r} dp^{\sigma_r}$.

ex) $[dV]_3 = (u_1 \wedge u_3 \wedge u_4) dp^1 dp^3 dp^4$

D. $\partial[dV]_r$, boundary of $[dV]_r$

The 2r (r-1)-vector boundaries on the surface of $[dV]_r$ are $\partial[dV]_r$.

$$\partial[dV]_r = \frac{\pm u^i}{dp^i} \cdot [dV]_r = \begin{cases} \frac{+u^i}{dp^i} \cdot [dV]_r & at\ P(\dots, p^i + dp^i, \dots) \\ \frac{-u^i}{dp^i} \cdot [dV]_r & at\ P(\dots, p^i, \dots) \end{cases}$$

E. examples of $\partial[dV]_r$

ex) $[dV]_1 = u_2 dp^2$

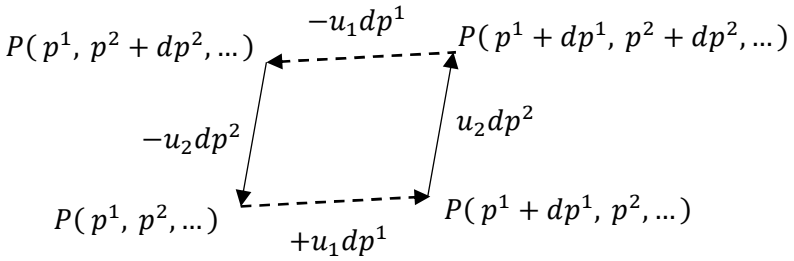
$$\partial[dV]_1 = \frac{\pm u^i}{dp^i} \cdot [dV]_1 = \begin{cases} \frac{+u^2}{dp^2} \cdot [dV]_1 & at\ P(\dots, p^2 + dp^2, \dots) \\ \frac{-u^2}{dp^2} \cdot [dV]_1 & at\ P(\dots, p^2, \dots) \end{cases}$$

$$\underset{-1}{P(\dots, p^2, \dots)} \xrightarrow[u_2 dp^2]{} \underset{+1}{P(\dots, p^2 + dp^2, \dots)}$$

$\partial[dV]_1$ is ± 1. It has opposite signs to adjacent $\partial[dV]_1$. They cancel out when integrated.

ex) $[dV]_2 = (u_1 \wedge u_2) dp^1 dp^2$

$$\partial[dV]_2 = \frac{\pm u^i}{dp^i} \cdot [dV]_2 = \begin{cases} \frac{+u^1}{dp^1} \cdot (u_1 \wedge u_2) dp^1 dp^2 = +u_2 dp^2 & at\ P(\dots, p^1 + dp^1, \dots) \\ \frac{-u^1}{dp^1} \cdot (u_1 \wedge u_2) dp^1 dp^2 = -u_2 dp^2 & at\ P(\dots, p^1, \dots) \\ \frac{+u^2}{dp^2} \cdot (u_1 \wedge u_2) dp^1 dp^2 = -u_1 dp^1 & at\ P(\dots, p^2 + dp^2, \dots) \\ \frac{-u^2}{dp^2} \cdot (u_1 \wedge u_2) dp^1 dp^2 = +u_1 dp^1 & at\ P(\dots, p^2, \dots) \end{cases}$$



$\partial[dV]_2$ rotates in one direction, so it is curled. It rotates in the opposite direction to other adjacent $\partial[dV]_2$. When integrated, they cancel out. Stokes's theorem can be derived.

3 curl

r-vector field A in n-dimensional.

ex) 3-vector field $A = A_{ijk}(u^i \wedge u^j \wedge u^k),\ (i < j < k)$

The definition of $curl(A)$ is the formula itself.

$$A \cdot \partial[dV]_r = A \cdot \left(\frac{\pm u^i}{dp^i} \cdot [dV]_r\right) = \left(\frac{\pm u^i}{dp^i} \wedge A\right) \cdot [dV]_r = \left(u^i \wedge \frac{A}{\partial p^i}\right) \cdot [dV]_r$$

$$= curl(A) \cdot [dV]_r\,, \qquad curl(A) = u^i \wedge \frac{A}{\partial p^i}$$

If you integrate it, you get Stokes's theorem.

$$\oint A \cdot \partial[dV]_r = \oint curl(A) \cdot [dV]_r$$

In calculus, write a vector field in a reciprocal basis. Then, the basis cancels out with a dot product. Then, differentiate only the coefficients of A to find $curl(A)$. Cartesian coordinates already use a reciprocal basis. The covariant derivative is superfluous.

A. $curl^2(A) = 0$

$$curl^2(A) = curl\{curl(A)\}$$

$$= u^j \wedge u^i \wedge \frac{\partial^2 A}{\partial p^j \partial p^i} = u^i \wedge u^j \wedge \frac{\partial^2 A}{\partial p^i \partial p^j} = -u^j \wedge u^i \wedge \frac{\partial^2 A}{\partial p^j \partial p^i} = 0$$

If both A and $curl(A)$ are not 0, then there is no $curl^2(A)$, but there is also no B such that $curl(B) = A$. $curl^2(B) = 0 = curl(A) \neq 0$. A and $curl(A)$ are one entity spanning two dimensions.

B. example of curl

Let's check the definition of $curl$ with the well-known vector field $A = A_r \mathbb{e}_r + A_\theta \mathbb{e}_\theta + A_\emptyset \mathbb{e}_\emptyset$ in spherical coordinates. To do so, first convert $\mathbb{e}_r, \mathbb{e}_\theta, \mathbb{e}_\emptyset$ to curvilinear coordinates basis $u^r, u^\theta, u^\emptyset$, calculate it, and then convert the result back to $\mathbb{e}_r, \mathbb{e}_\theta, \mathbb{e}_\emptyset$ and check it.

$$p^1 = r, \qquad p^2 = \theta, \qquad p^3 = \emptyset$$

$$u_r = \frac{\partial P}{\partial r} = \mathbb{e}_r, \qquad u_\theta = \frac{\partial P}{\partial \theta} = r\mathbb{e}_\theta, \qquad u_\emptyset = \frac{\partial P}{\partial \emptyset} = r\sin\theta\, \mathbb{e}_\emptyset$$

$$u^r \cdot u_r = u^\theta \cdot u_\theta = u^\emptyset \cdot u_\emptyset = 1, \qquad u^r = \mathbb{e}_r, \qquad u^\theta = \frac{\mathbb{e}_\theta}{r}, \qquad u^\emptyset = \frac{\mathbb{e}_\emptyset}{r\sin\theta}$$

$$A = A_r \mathbb{e}_r + A_\theta \mathbb{e}_\theta + A_\emptyset \mathbb{e}_\emptyset = A_r u^r + A_\theta r u^\theta + A_\emptyset r \sin\theta \, u^\emptyset$$

Since there are many terms, let's calculate only the $\left(u^\theta \wedge u^\emptyset\right)$ term.

$$curl(A) = u^i \wedge \frac{A}{\partial p^i}$$

$$u^\theta \wedge \frac{\partial(A_\emptyset r \sin\theta)}{\partial\theta} u^\emptyset + u^\emptyset \wedge \frac{\partial(A_\theta r)}{\partial\emptyset} u^\theta = \left\{\frac{\partial(A_\emptyset r \sin\theta)}{\partial\theta} - \frac{\partial(A_\theta r)}{\partial\emptyset}\right\}\left(u^\theta \wedge u^\emptyset\right)$$

$$= \frac{1}{r^2 \sin\theta}\left\{\frac{\partial(A_\emptyset r \sin\theta)}{\partial\theta} - \frac{\partial(A_\theta r)}{\partial\emptyset}\right\}(\mathbb{e}_\theta \wedge \mathbb{e}_\emptyset)$$

Compared to the conventional results, the coefficients are the same, However, it is not the $\mathbb{e}_r$ term, but the $(\mathbb{e}_\theta \wedge \mathbb{e}_\emptyset)$ term. These two are dual basis each other. Fortunately, we were unable to distinguish.

C. divergence

$divergence(A)$ is $curl$ with respect to A^d.

$$div(A) = curl\left(A^d\right), \qquad A^d = A \cdot I$$

D. example of divergence

Let us take the same vector field $A = A_r \mathbb{e}_r + A_\theta \mathbb{e}_\theta + A_\emptyset \mathbb{e}_\emptyset$ as an example. In three dimensions,

$$A = A_r \mathbb{e}_r + A_\theta \mathbb{e}_\theta + A_\emptyset \mathbb{e}_\emptyset = A_r u_r + \frac{A_\theta}{r} u_\theta + \frac{A_\emptyset}{r \sin\theta} u_\emptyset$$

$$I = \mathbb{e}_1 \wedge \mathbb{e}_2 \wedge \mathbb{e}_3 = \mathbb{e}_r \wedge \mathbb{e}_\theta \wedge \mathbb{e}_\emptyset = r^2 \sin\theta \left(u^r \wedge u^\theta \wedge u^\emptyset\right)$$

$$\mathbb{e}_r = \cos\theta \, \mathbb{e}_3 + r \sin\theta \cos\emptyset \, \mathbb{e}_1 + r \sin\theta \sin\emptyset \, \mathbb{e}_2 = u^r$$

$$\mathbb{e}_\theta = \cdots = r u^\theta, \qquad \mathbb{e}_\emptyset = \cdots = r \sin\theta \, u^\emptyset$$

$$A^d = A \cdot I = \left(A_r u_r + \frac{A_\theta}{r} u_\theta + \frac{A_\emptyset}{r \sin\theta} u_\emptyset\right) \cdot r^2 \sin\theta \left(u^r \wedge u^\theta \wedge u^\emptyset\right)$$

$$= A_r r^2 \sin\theta \left(u^\theta \wedge u^\emptyset\right) - A_\theta r \sin\theta \left(u^r \wedge u^\emptyset\right) + A_\emptyset r \left(u^r \wedge u^\theta\right)$$

$$div(A) = curl\left(A^d\right) = u^i \wedge \frac{\partial A^d}{\partial p^i}$$

$$= \left\{\frac{\partial(A_r r^2 \sin\theta)}{\partial r} + \frac{\partial(A_\theta r \sin\theta)}{\partial\theta} + \frac{\partial(A_\emptyset r)}{\partial\emptyset}\right\}\left(u^r \wedge u^\theta \wedge u^\emptyset\right)$$

$$= \frac{1}{r^2 \sin\theta}\left\{\frac{\partial(A_r r^2 \sin\theta)}{\partial r} + \frac{\partial(A_\theta r \sin\theta)}{\partial\theta} + \frac{\partial(A_\emptyset r)}{\partial\emptyset}\right\}(\mathbb{e}_r \wedge \mathbb{e}_\theta \wedge \mathbb{e}_\emptyset)$$

Compared to the conventional results, the coefficients are the same, However, it is not a scalar term but a $I = (\mathbb{e}_r \wedge \mathbb{e}_\theta \wedge \mathbb{e}_\emptyset)$ term. These two are also dual basis each other.

E. $div(A) = 0$

$$div(A) \cdot [dV]_r = curl\left(A^d\right) \cdot [dV]_r = A^d \cdot \partial[dV]_r = (A \cdot I) \cdot \partial[dV]_r$$

$$= (A \wedge \partial[dV]_r) \cdot I$$

Although not strictly accurate, $(A \wedge \partial[dV]_r) \cdot I$ represents the magnitude of a physical quantity that flows in and out through $\partial[dV]_r$ by a speed of A.

$A \wedge \partial[dV]_r$ is multiplied only by the A components perpendicular to $\partial[dV]_r$ and the A components parallel to $\partial[dV]_r$ cancel out through the wedge product $(\wedge)$. The dot product of I with this is $div(A) \cdot [dV]_r$.

If $div(A) = 0$ at all points, this means that a physical quantity flows without accumulation or leakage. In this case, A is said to be continuous.

4 four vector fields

A. postulate

For a vector field A to have physical meaning, the following conditions are required:

1) A vector field is a single entity spanning two dimensions. A and $curl(A)$ must exist.

2) Vector field A is continuous. $div(A) = 0$.

3) $curl^d(A)$ also exists. Duality. It is a dual-dimensional entity for $curl(A)$. It is not a new entity, but rather the other side of the same coin.

4) And for the same reason as 1), $curl\{curl^d(A)\} = div\{curl(A)\} = Lap(A) = J$ also exists. Lap stands for Laplacian.

5) $curl^d(A)$ is also continuous. The sign varies depending on the dimension.
$div\{curl^d(A)\} = curl\{curl^d(A)\}^d = \pm curl^2(A) = 0.$

6) The following two equations are properties of $curl$.
$curl^2(A) = 0,$ $curl(J) = curl\{Lap(A)\} = curl^2\{curl^d(A)\} = 0$

C. possible four vector fields

There are four vector fields that satisfy the above assumptions in four dimensions, including time. For readability, different fonts are used for each dimension.

1-vector field A

$$A = A_1\mathbb{e}^1 + A_2\mathbb{e}^2 + A_3\mathbb{e}^3 + A_t\mathbb{e}^t$$

2-vector field $(\mathcal{B} + \mathcal{E})$

$$\mathcal{B} + \mathcal{E} = \mathcal{B}_1(\mathbb{e}^2 \wedge \mathbb{e}^3) + \mathcal{B}_2(\mathbb{e}^3 \wedge \mathbb{e}^1) + \mathcal{B}_3(\mathbb{e}^1 \wedge \mathbb{e}^2)$$
$$+\mathcal{E}_1(\mathbb{e}^1 \wedge \mathbb{e}^t) + \mathcal{E}_2(\mathbb{e}^2 \wedge \mathbb{e}^t) + \mathcal{E}_3(\mathbb{e}^3 \wedge \mathbb{e}^t)$$

3-vector field $\mathbb{A}$

$$\mathbb{A} = \mathbb{A}_1(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + \mathbb{A}_2(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) + \mathbb{A}_3(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t) + \mathbb{A}_t(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)$$

scalar field $\mathscr{v}$

$$\mathscr{v}$$

Using only the above conditions, we derive Maxwell's equations for the 1-vector field A. We repeat the same process for vector fields of other dimensions and interpret the results.

B. Minkowski space

It is written in Cartesian coordinates. $P(x^1, x^2, x^3, x^4 = t)$. The relationship between the basis $\mathbb{e}_1, \mathbb{e}_2, \mathbb{e}_3, \mathbb{e}_t$ and the reciprocal basis $\mathbb{e}^1, \mathbb{e}^2, \mathbb{e}^3, \mathbb{e}^t$ is as follows.

$$\mathbb{e}_i = \frac{\partial P}{\partial x^i}, \qquad \mathbb{e}_4 = \frac{\partial P}{\partial x^4} = \frac{\partial P}{\partial t} = \mathbb{e}_t, \qquad i, j = 1,2,3$$

$$\mathbb{e}_i \cdot \mathbb{e}_j = \delta_{ij}, \qquad \mathbb{e}^i \cdot \mathbb{e}_j = \delta^i_j, \qquad \mathbb{e}^i = \mathbb{e}_i$$

$$\mathbb{e}_t \cdot \mathbb{e}_t = \mathbb{e}^t \cdot \mathbb{e}^t = -1, \qquad \mathbb{e}^t \cdot \mathbb{e}_t = 1, \qquad \mathbb{e}^t = -\mathbb{e}_t$$

unit pseudoscalar $I = \mathbb{e}_1 \wedge \mathbb{e}_2 \wedge \mathbb{e}_3 \wedge \mathbb{e}_t = -\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t$

Partial differentiation notation

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \qquad \partial_{\mu\nu} = \frac{\partial^2}{\partial x^\mu \partial x^\nu}, \qquad \mu, \nu = 1,2,3,4$$

$$\square = \partial_{tt} - \partial_{11} - \partial_{22} - \partial_{33}, \qquad d'Alembert\ operator$$

4.1 1-vector field

Once again, it is better to express the vector field in a reciprocal basis.

$$A = A_1\mathbb{e}^1 + A_2\mathbb{e}^2 + A_3\mathbb{e}^3 + A_t\mathbb{e}^t = A^1\mathbb{e}_1 + A^2\mathbb{e}_2 + A^3\mathbb{e}_3 + A^t\mathbb{e}_t$$

$$A_i = A^i, \qquad A_t = -A^t = -\varphi$$

A. $curl(A)$

$$curl(A) = \mathbb{e}^\mu \wedge \partial_\mu A = B + E, \qquad \mu = 1,2,3,4$$

$$= (\partial_2 A_3 - \partial_3 A_2)(\mathbb{e}^2 \wedge \mathbb{e}^3) + (\partial_3 A_1 - \partial_1 A_3)(\mathbb{e}^3 \wedge \mathbb{e}^1) + (\partial_1 A_2 - \partial_2 A_1)(\mathbb{e}^1 \wedge \mathbb{e}^2)$$

$$+(\partial_1 A_t - \partial_t A_1)(\mathbb{e}^1 \wedge \mathbb{e}^t) + (\partial_2 A_t - \partial_t A_2)(\mathbb{e}^2 \wedge \mathbb{e}^t) + (\partial_3 A_t - \partial_t A_3)(\mathbb{e}^3 \wedge \mathbb{e}^t)$$

$$= B_1(\mathbb{e}^2 \wedge \mathbb{e}^3) + B_2(\mathbb{e}^3 \wedge \mathbb{e}^1) + B_3(\mathbb{e}^1 \wedge \mathbb{e}^2) + E_1(\mathbb{e}^1 \wedge \mathbb{e}^t) + E_2(\mathbb{e}^2 \wedge \mathbb{e}^t) + E_3(\mathbb{e}^3 \wedge \mathbb{e}^t)$$

$$B: magnetic\ field, \qquad E: elctric\ field$$

B. $curl^2(A) = 0$

$$curl^2(A) = \mathbb{e}^\mu \wedge \partial_\mu\{curl(A)\}$$

$$= (\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3)(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) + (\partial_2 E_3 - \partial_3 E_2 + \partial_t B_1)(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)$$

$$+(\partial_3 E_1 - \partial_1 E_3 + \partial_t B_2)(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) + (\partial_1 E_2 - \partial_2 E_1 + \partial_t B_3)(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t) = 0$$

Faraday's law, the conventional representation, but it is in 3-vector terms.

$$\nabla \cdot B = 0, \qquad \nabla \times E + \partial_t B = 0$$

C. $div(A) = 0$

$$A^d = A \cdot I = -A_1(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) - A_2(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) - A_1(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t) - A_t(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)$$

$$div(A) = curl\{A^d\} = \mathbb{e}^\mu \wedge \partial_\mu A^d = (\partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 - \partial_t A_t)I$$

Lorentz condition, but it is in pseudoscalar terms.

$$\nabla \cdot A + \partial_t \varphi = 0$$

D. $Lap(A) = J$

$$curl^d(A) = curl(A) \cdot I$$

$$= E_1(\mathbb{e}^2 \wedge \mathbb{e}^3) + E_2(\mathbb{e}^3 \wedge \mathbb{e}^1) + E_3(\mathbb{e}^1 \wedge \mathbb{e}^2) - B_1(\mathbb{e}^1 \wedge \mathbb{e}^t) - B_2(\mathbb{e}^2 \wedge \mathbb{e}^t) - B_3(\mathbb{e}^3 \wedge \mathbb{e}^t)$$

$$Lap(A) = \mathbb{e}^\mu \wedge \partial_\mu\{curl^d(A)\} = J$$

$$= (\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3)(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) - (\partial_2 B_3 - \partial_3 B_2 - \partial_t E_1)(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)$$

$$-(\partial_3 B_1 - \partial_1 B_3 - \partial_t E_2)(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) - (\partial_1 B_2 - \partial_2 B_1 - \partial_t E_3)(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t)$$

$$= -J_t(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) - J_1(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) - J_2(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) - J_3(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t)$$

$$J_t = -\rho: charge\ density, \qquad J_i: current\ density$$

Gauss's law and Ampère–Maxwell law, which are 3-vector terms.

$$\nabla \cdot E = -J_t = \rho, \qquad \nabla \times B = J + \partial_t E$$

E. particle and wave

Let's substitute A_μ for E_i, B_j in the previous equations.（4.1A and 4.1C）

$$-J_t = \partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3 = \partial_{11} A_t - \partial_{1t} A_1 + \partial_{22} A_t - \partial_{2t} A_2 + \partial_{33} A_t - \partial_{3t} A_3$$

$$= \partial_{11} A_t + \partial_{22} A_t + \partial_{33} A_t - \partial_{tt} A_t - \partial_t(\partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 - \partial_t A_t)$$

$$= \partial_{11} A_t + \partial_{22} A_t + \partial_{33} A_t - \partial_{tt} A_t = -\Box A_t$$

$$J_1 = \partial_2 B_3 - \partial_3 B_2 - \partial_t E_1 = \partial_{21} A_2 - \partial_{22} A_1 - \partial_{33} A_1 + \partial_{31} A_3 + \partial_{tt} A_1 - \partial_{t1} A_t$$

$$= \partial_{tt} A_1 - \partial_{11} A_1 - \partial_{22} A_1 - \partial_{33} A_1 + \partial_1(\partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 - \partial_t A_t) = \Box A_1$$

$$J_t = \Box A_t \qquad J_1 = \Box A_1 \qquad J_2 = \Box A_2 \qquad J_3 = \Box A_3$$

F. $curl(J) = 0$

$$curl(J) = \mathbb{e}^\mu \wedge \partial_\mu J = (\partial_1 J_1 + \partial_2 J_2 + \partial_3 J_3 - \partial_t J_t) I = 0$$

Conservative law, pseudoscalar term.

$$\nabla \cdot J + \partial_t \rho = 0$$

Now, let's expand the dimension of the vector field.

4.2 2-vector field

$$\mathcal{B}+\mathcal{E}=\mathcal{B}_1(\mathbb{e}^2\wedge\mathbb{e}^3)+\mathcal{B}_2(\mathbb{e}^3\wedge\mathbb{e}^1)+\mathcal{B}_3(\mathbb{e}^1\wedge\mathbb{e}^2)$$
$$+\mathcal{E}_1(\mathbb{e}^1\wedge\mathbb{e}^t)+\mathcal{E}_2(\mathbb{e}^2\wedge\mathbb{e}^t)+\mathcal{E}_3(\mathbb{e}^3\wedge\mathbb{e}^t)$$

A. $curl(\mathcal{B}+\mathcal{E})$

$$curl(\mathcal{B}+\mathcal{E})=\mathbb{e}^\mu\wedge\partial_\mu(\mathcal{B}+\mathcal{E})=\mathcal{A}$$
$$=\mathcal{A}_t(\mathbb{e}^1\wedge\mathbb{e}^2\wedge\mathbb{e}^3)+\mathcal{A}_1(\mathbb{e}^2\wedge\mathbb{e}^3\wedge\mathbb{e}^t)+\mathcal{A}_2(\mathbb{e}^3\wedge\mathbb{e}^1\wedge\mathbb{e}^t)+\mathcal{A}_3(\mathbb{e}^1\wedge\mathbb{e}^2\wedge\mathbb{e}^t)$$
$$\mathcal{A}_t=\partial_1\mathcal{B}_1+\partial_2\mathcal{B}_2+\partial_3\mathcal{B}_3,\qquad \mathcal{A}_1=\partial_2\mathcal{E}_3-\partial_3\mathcal{E}_2+\partial_t\mathcal{B}_1$$
$$\mathcal{A}_2=\partial_3\mathcal{E}_1-\partial_1\mathcal{E}_3+\partial_t\mathcal{B}_2,\qquad \mathcal{A}_3=\partial_1\mathcal{E}_2-\partial_2\mathcal{E}_1+\partial_t\mathcal{B}_3$$

Conventional expression

$$\mathcal{A}_t=\nabla\cdot\mathcal{B},\qquad \mathcal{A}=\nabla\times\mathcal{E}+\partial_t\mathcal{B}$$

B. $curl^2(\mathcal{B}+\mathcal{E})=0$

$$curl^2(\mathcal{B}+\mathcal{E})=\mathbb{e}^\mu\wedge\partial_\mu\mathcal{A}=(\partial_1\mathcal{A}_1+\partial_2\mathcal{A}_2+\partial_3\mathcal{A}_3-\partial_t\mathcal{A}_t)(-I)=0$$

Conventional expression

$$\nabla\cdot\mathcal{A}-\partial_t\mathcal{A}_t=0$$

C. $div(\mathcal{B}+\mathcal{E})=0$

$$(\mathcal{B}+\mathcal{E})^d=(\mathcal{B}+\mathcal{E})\cdot I$$
$$=\mathcal{E}_1(\mathbb{e}^2\wedge\mathbb{e}^3)+\mathcal{E}_2(\mathbb{e}^3\wedge\mathbb{e}^1)+\mathcal{E}_3(\mathbb{e}^2\wedge\mathbb{e}^3)-\mathcal{B}_1(\mathbb{e}^1\wedge\mathbb{e}^t)-\mathcal{B}_2(\mathbb{e}^2\wedge\mathbb{e}^t)-\mathcal{B}_3(\mathbb{e}^3\wedge\mathbb{e}^t)$$
$$div(\mathcal{B}+\mathcal{E})=curl\{\mathcal{B}^d+\mathcal{E}^d\}$$
$$=(\partial_1\mathcal{E}_1+\partial_2\mathcal{E}_2+\partial_3\mathcal{E}_3)(\mathbb{e}^1\wedge\mathbb{e}^2\wedge\mathbb{e}^3)-(\partial_2\mathcal{B}_3-\partial_3\mathcal{B}_2-\partial_t\mathcal{E}_1)(\mathbb{e}^2\wedge\mathbb{e}^3\wedge\mathbb{e}^t)$$
$$-(\partial_3\mathcal{B}_1-\partial_1\mathcal{B}_3-\partial_t\mathcal{E}_2)(\mathbb{e}^3\wedge\mathbb{e}^1\wedge\mathbb{e}^t)-(\partial_1\mathcal{B}_2-\partial_2\mathcal{B}_1-\partial_t\mathcal{E}_3)(\mathbb{e}^1\wedge\mathbb{e}^2\wedge\mathbb{e}^t)=0$$

Conventional expression

$$\nabla\cdot\mathcal{E}=0,\qquad \nabla\times\mathcal{B}-\partial_t\mathcal{E}=0$$

D. $Lap(\mathcal{B}+\mathcal{E})=\mathcal{J}$

$$curl^d(\mathcal{B}+\mathcal{E})=curl(\mathcal{B}+\mathcal{E})\cdot I=-\mathcal{A}_1\mathbb{e}^1-\mathcal{A}_2\mathbb{e}^2-\mathcal{A}_3\mathbb{e}^3-\mathcal{A}_t\mathbb{e}^t$$
$$Lap(\mathcal{B}+\mathcal{E})=\mathbb{e}^\mu\wedge\partial_\mu\{curl^d(\mathcal{B}+\mathcal{E})\}=\mathcal{J}$$
$$=(\partial_3\mathcal{A}_2-\partial_2\mathcal{A}_3)(\mathbb{e}^2\wedge\mathbb{e}^3)+(\partial_1\mathcal{A}_3-\partial_3\mathcal{A}_1)(\mathbb{e}^3\wedge\mathbb{e}^1)+(\partial_2\mathcal{A}_1-\partial_1\mathcal{A}_2)(\mathbb{e}^1\wedge\mathbb{e}^2)$$

$$+(\partial_t\mathcal{A}_1-\partial_1\mathcal{A}_t)(\mathbb{e}^1\wedge\mathbb{e}^t)+(\partial_t\mathcal{A}_2-\partial_2\mathcal{A}_t)(\mathbb{e}^2\wedge\mathbb{e}^t)+(\partial_t\mathcal{A}_3-\partial_3\mathcal{A}_t)(\mathbb{e}^3\wedge\mathbb{e}^t)$$

$$=-\mathcal{K}_1(\mathbb{e}^2\wedge\mathbb{e}^3)-\mathcal{K}_2(\mathbb{e}^3\wedge\mathbb{e}^1)-\mathcal{K}_3(\mathbb{e}^1\wedge\mathbb{e}^2)+\mathcal{L}_1(\mathbb{e}^1\wedge\mathbb{e}^t)+\mathcal{L}_2(\mathbb{e}^2\wedge\mathbb{e}^t)+\mathcal{L}_3(\mathbb{e}^3\wedge\mathbb{e}^t)$$

$\mathcal{K}_i$: *spin density*, $\mathcal{L}_i$: *spin current density*

Conventional expression

$$-\mathcal{K}=\nabla\times\mathcal{A},\qquad -\mathcal{L}=\nabla\mathcal{A}_t-\partial_t\mathcal{A}$$

E. particle and wave

Let's substitute $\mathcal{A}_\mu$ for $\mathcal{B}_i,\mathcal{E}_j$ in the previous equations.（4.2A and 4.2C）

$$\mathcal{K}_1=\partial_2\mathcal{A}_3-\partial_3\mathcal{A}_2=\partial_2(\partial_1\mathcal{E}_2-\partial_2\mathcal{E}_1+\partial_t\mathcal{B}_3)-\partial_3(\partial_3\mathcal{E}_1-\partial_1\mathcal{E}_3+\partial_t\mathcal{B}_2)$$

$$=-\partial_{22}\mathcal{E}_1-\partial_{33}\mathcal{E}_1+\partial_t(\partial_2\mathcal{B}_3-\partial_3\mathcal{B}_2)+\partial_1(\partial_2\mathcal{E}_2+\partial_3\mathcal{E}_3)$$

$$=-\partial_{22}\mathcal{E}_1-\partial_{33}\mathcal{E}_1+\partial_t(\partial_t\mathcal{E}_1)-\partial_1(\partial_1\mathcal{E}_1)=\Box\mathcal{E}_1,\qquad \nabla\cdot\mathcal{E}=0,\qquad \nabla\times\mathcal{B}-\partial_t\mathcal{E}=0$$

$$\mathcal{K}_1=\Box\mathcal{E}_1\quad \mathcal{K}_2=\Box\mathcal{E}_2\quad \mathcal{K}_3=\Box\mathcal{E}_3$$

$$\mathcal{L}_1=-(\partial_1\mathcal{A}_t-\partial_t\mathcal{A}_1)=-\partial_1(\partial_1\mathcal{B}_1+\partial_2\mathcal{B}_2+\partial_3\mathcal{B}_3)+\partial_t(\partial_2\mathcal{E}_3-\partial_3\mathcal{E}_2+\partial_t\mathcal{B}_1)$$

$$=-\partial_{11}\mathcal{B}_1+\partial_{tt}\mathcal{B}_1-\partial_2(\partial_1\mathcal{B}_2-\partial_t\mathcal{E}_3)-\partial_3(\partial_1\mathcal{B}_3+\partial_t\mathcal{E}_2)$$

$$=-\partial_{11}\mathcal{B}_1+\partial_{tt}\mathcal{B}_1-\partial_2(\partial_2\mathcal{B}_1)-\partial_3(\partial_3\mathcal{B}_1)=\Box\mathcal{B}_1,$$

$$\mathcal{L}_1=\Box\mathcal{B}_1\quad \mathcal{L}_2=\Box\mathcal{B}_2\quad \mathcal{L}_3=\Box\mathcal{B}_3$$

F. $curl(\mathcal{J})=0$

$$curl(\mathcal{J})=curl(\mathcal{K}+\mathcal{L})=\mathbb{e}^\mu\wedge\partial_\mu(\mathcal{K}+\mathcal{L})$$

$$=-(\partial_1\mathcal{K}_1+\partial_2\mathcal{K}_2+\partial_3\mathcal{K}_3)(\mathbb{e}^1\wedge\mathbb{e}^2\wedge\mathbb{e}^3)+(\partial_2\mathcal{L}_3-\partial_3\mathcal{L}_2-\partial_t\mathcal{K}_1)(\mathbb{e}^2\wedge\mathbb{e}^3\wedge\mathbb{e}^t)$$

$$+(\partial_3\mathcal{L}_1-\partial_1\mathcal{L}_3-\partial_t\mathcal{K}_2)(\mathbb{e}^3\wedge\mathbb{e}^1\wedge\mathbb{e}^t)+(\partial_1\mathcal{L}_2-\partial_2\mathcal{L}_1-\partial_t\mathcal{K}_3)(\mathbb{e}^1\wedge\mathbb{e}^2\wedge\mathbb{e}^t)=0$$

Conventional expression

$$\nabla\cdot\mathcal{K}=0\qquad \nabla\times\mathcal{L}-\partial_t\mathcal{K}=0$$

4.3 3-vector field

A. $curl(\mathbb{A})$

$$\mathbb{A} = -\mathbb{A}_1 e^2 \wedge e^3 \wedge e^t - \mathbb{A}_2 e^3 \wedge e^1 \wedge e^t - \mathbb{A}_3 e^1 \wedge e^2 \wedge e^t - \mathbb{A}_t e^1 \wedge e^2 \wedge e^3$$

$$curl(\mathbb{A}) = e^\mu \wedge \partial_\mu \mathbb{A} = \mathbb{V} I \qquad \mathbb{V} = -\partial_t \mathbb{A}_t + \partial_1 \mathbb{A}_1 + \partial_2 \mathbb{A}_2 + \partial_3 \mathbb{A}_3$$

$$curl^2(\mathbb{A}) = e^\mu \wedge \partial_\mu \mathbb{V} = null$$

C. $div(\mathbb{A}) = 0$

$$\mathbb{A}^d = \mathbb{A} \cdot I = \mathbb{A}_1 e^1 + \mathbb{A}_2 e^2 + \mathbb{A}_3 e^3 + \mathbb{A}_t e^t$$

$$div(\mathbb{A}) = curl\left(\mathbb{A}^d\right) = e^\mu \wedge \partial_\mu \mathbb{A}^d$$

$$= (\partial_2 \mathbb{A}_3 - \partial_3 \mathbb{A}_2)(e^2 \wedge e^3) + (\partial_3 \mathbb{A}_1 - \partial_1 \mathbb{A}_3)(e^3 \wedge e^1) + (\partial_1 \mathbb{A}_2 - \partial_2 \mathbb{A}_1)(e^1 \wedge e^2)$$

$$+(\partial_1 \mathbb{A}_t - \partial_t \mathbb{A}_1)(e^1 \wedge e^t) + (\partial_2 \mathbb{A}_t - \partial_t \mathbb{A}_2)(e^2 \wedge e^t) + (\partial_3 \mathbb{A}_t - \partial_t \mathbb{A}_3)(e^3 \wedge e^t) = 0$$

Conventional expression, $\nabla \times \mathbb{A} = 0 \qquad \nabla \mathbb{A}_t - \partial_t \mathbb{A} = 0$

D. $Lap(\mathbb{A}) = \mathbb{J}$

$$curl^d(\mathbb{A}) = \mathbb{V} I^d = \mathbb{V} \qquad Lap(\mathbb{A}) = e^\mu \wedge \partial_\mu \{curl^d(\mathbb{A})\} = \mathbb{J}$$

$$= \partial_1 \mathbb{V} e^1 + \partial_2 \mathbb{V} e^2 + \partial_3 \mathbb{V} e^3 + \partial_t \mathbb{V} e^t = \mathbb{J}_1 e^1 + \mathbb{J}_2 e^2 + \mathbb{J}_3 e^3 + \mathbb{J}_t e^t$$

$\mathbb{J}_i$: *color density*, $\mathbb{J}_t$: *color current density*

Conventional expression, $\mathbb{J} = \nabla \mathbb{V} \qquad \mathbb{J}_t = \partial_t \mathbb{V}$

E. particle and wave

Let's substitute $\mathbb{A}_\mu$ for $\mathbb{V}$ in the previous equations.（4.3A and 4.3C）

$$\mathbb{J}_1 = \partial_1 \mathbb{V} = \partial_1(\partial_t \mathbb{A}_t - \partial_1 \mathbb{A}_1 - \partial_2 \mathbb{A}_2 - \partial_3 \mathbb{A}_3) = \partial_{t1} \mathbb{A}_t - \partial_{11} \mathbb{A}_1 - \partial_{21} \mathbb{A}_2 - \partial_{31} \mathbb{A}_3$$

$$= \partial_{tt} \mathbb{A}_1 - \partial_{11} \mathbb{A}_1 - \partial_{22} \mathbb{A}_1 - \partial_{33} \mathbb{A}_1 = \Box \mathbb{A}_1$$

$$\mathbb{J}_t = \partial_t \mathbb{V} = \partial_t(\partial_t \mathbb{A}_t - \partial_1 \mathbb{A}_1 - \partial_2 \mathbb{A}_2 - \partial_3 \mathbb{A}_3) = \partial_{tt} \mathbb{A}_t - \partial_{1t} \mathbb{A}_1 - \partial_{2t} \mathbb{A}_2 - \partial_{3t} \mathbb{A}_t$$

$$= \partial_{tt} A_t - \partial_{11} \mathbb{A}_t - \partial_{22} \mathbb{A}_t - \partial_{33} \mathbb{A}_t = \Box \mathbb{A}_t$$

$$\mathbb{J}_1 = \Box \mathbb{A}_1 \qquad \mathbb{J}_2 = \Box \mathbb{A}_2 \qquad \mathbb{J}_3 = \Box \mathbb{A}_3 \qquad \mathbb{J}_t = \Box \mathbb{A}_t$$

F. $curl(\mathbb{J}) = 0$

$$curl(\mathbb{J}) = e^\mu \wedge \partial_\mu(\mathbb{J}) = 0$$

$$(\partial_2 \mathbb{J}_3 - \partial_3 \mathbb{J}_2)(e^2 \wedge e^3) = (\partial_3 \mathbb{J}_1 - \partial_1 \mathbb{J}_3)(e^3 \wedge e^1) = (\partial_1 \mathbb{J}_2 - \partial_2 \mathbb{J}_1)(e^1 \wedge e^2) = 0$$

$$(\partial_1 \mathbb{J}_t - \partial_t \mathbb{J}_1)(e^1 \wedge e^t) = (\partial_2 \mathbb{J}_t - \partial_t \mathbb{J}_2)(e^2 \wedge e^t) = (\partial_3 \mathbb{J}_t - \partial_t \mathbb{J}_3)(e^3 \wedge e^t) = 0$$

Conventional expression, $\nabla \times \mathbb{J} = 0 \qquad \nabla \mathbb{J}_t - \partial_t \mathbb{J} = 0$

4.4 scalar field v

A. $curl(v)$

$curl(v) = e^{\mu} \wedge \partial_{\mu} v = a \quad a = a_1 e^1 + a_2 e^2 + a_3 e^3 + a_t e^t$

$a_1 = \partial_1 v, \quad a_2 = \partial_2 v, \quad a_3 = \partial_3 v, \quad a_t = \partial_t v$

B. $curl^2(v) = 0$

$curl^2(v) = e^{\mu} \wedge \partial_{\mu} a = 0$

$(\partial_2 a_3 - \partial_3 a_2)(e^2 \wedge e^3) + (\partial_3 a_1 - \partial_1 a_3)(e^3 \wedge e^1) + (\partial_1 a_2 - \partial_2 a_1)(e^1 \wedge e^2) +$

$(\partial_1 a_t - \partial_t a_1)(e^1 \wedge e^t) + (\partial_2 a_t - \partial_t a_2)(e^2 \wedge e^t) + (\partial_3 a_t - \partial_t a_3)(e^3 \wedge e^t) = 0$

Conventional expression, $\nabla \times a = 0 \quad \nabla a_t - \partial_t a = 0$

C. $div(v) = null$

$v^d = v 1^d = vI$

$div(v) = curl(v^d) = e^{\mu} \wedge \partial_{\mu} vI = null$

D. $Lap(v) = j$

$curl^d(v) = curl(v) \cdot I$

$= -a_t(e^1 \wedge e^2 \wedge e^3) - a_1(e^2 \wedge e^3 \wedge e^t) - a_2(e^3 \wedge e^1 \wedge e^t) - a_3(e^1 \wedge e^2 \wedge e^t)$

$Lap(v) = e^{\mu} \wedge \partial_{\mu}\{curl^d(v)\} = (-\partial_t a_t + \partial_1 a_1 + \partial_2 a_2 + \partial_3 a_3)I = -mI$

$m = \partial_t a_t - \nabla \cdot a, \quad m: mass\ density$

E. particle and wave

Let's substitute a_{μ} for v in the previous equations.（4.4A）

$m = \partial_t a_t - \nabla \cdot a = \partial_{tt} v - \partial_{11} v - \partial_{22} v - \partial_{33} v = \square v$

$m = \square v$

F. $curl(j) = null$

$curl(j) = e^{\mu} \wedge \partial_{\mu} j = e^{\mu} \wedge \partial_{\mu}(mI) = null$

4.5 questions

The vector field A has wave properties, and $J = Lapl(A)$ has particle properties. They are two aspects of a single entity. They are dual-dimensional. Duality.

If $J = div\{curl(A)\} \neq 0$, then $curl(A)$ is not continuous. A particle is a phenomenon in which $curl(A)$ accumulates or leaks out.

J varies depending on the dimension.

Color density	$:\mathbb{e}^1, \mathbb{e}^2, \mathbb{e}^3$	Spin density	$:(\mathbb{e}^2 \wedge \mathbb{e}^3), (\mathbb{e}^3 \wedge \mathbb{e}^1), (\mathbb{e}^1 \wedge \mathbb{e}^2)$
Charge density	$:(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)$	Mass density	$:(I = -\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)$

We can see the following:

1) Color is a concept of string.
2) Spin is a concept of membrane.
3) Charge is a particle concept.
4) The dimension of mass is pseudoscalar. It exists only over time.
5) If $\mathbb{e}^i$ is color (quark), then $\mathbb{e}^t$ is Higgs. It is essential for creating mass. 4.3E
6) The spin of an electron is not created by the rotation of charge.
7) From a charge perspective, it's no coincidence that protons contain three quarks. However, this does not mean that quarks have charge.

Conversely, some things are also unknown.

1) Are the spins $(\mathbb{e}^2 \wedge \mathbb{e}^3), (\mathbb{e}^3 \wedge \mathbb{e}^1), (\mathbb{e}^1 \wedge \mathbb{e}^2)$ is a subset of charge $(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)$?
2) Are positive and negative charges geometrically different (isomers)? $(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3), (\mathbb{e}^2 \wedge \mathbb{e}^1 \wedge \mathbb{e}^3)$
3) Are there protons and neutrons, but no negative ones?
4) Why is there no negative mass?
5) Electric current is the movement of charge, but if you look at the dimension of current density $\left(\mathbb{e}^i \wedge \mathbb{e}^j \wedge \mathbb{e}^t\right)$, a superconductor is a state. Perhaps time flows in a state where the spins are well aligned?

5 sedenion

In the next chapter, expressing energy density as a multi-vector requires a new definition of vector norm. This definition again defines multiplication in the 16 bases.

First, let's distinguish the basis because the characteristics of multiplication differ depending on the dimension.

$\mathbb{b}_1 = \{e^1, e^2, e^3, e^t\}$

$\mathbb{b}_2 = \{(e^1 \wedge e^2), (e^2 \wedge e^3), (e^3 \wedge e^1), (e^1 \wedge e^t), (e^2 \wedge e^t), (e^3 \wedge e^t)\}$

$\mathbb{b}_3 = \{(e^1 \wedge e^2 \wedge e^3), (e^1 \wedge e^2 \wedge e^t), (e^2 \wedge e^3 \wedge e^t), (e^3 \wedge e^1 \wedge e^t)\}$

sedenion Union of $1, \mathbb{b}_1, \mathbb{b}_2, \mathbb{b}_3, I$ $I = -(e^1 \wedge e^2 \wedge e^3 \wedge e^t)$

octonion Union of $1, \mathbb{b}_2, I$

quaternion $1, (e^1 \wedge e^2), (e^2 \wedge e^3), (e^3 \wedge e^1)$

This is not surprising. Pauli spin matrices are known to be related to quaternions.

A. vector norm

The existing definition of vector norm $\|v\| = v \cdot v$ is inconvenient. A new definition is needed.

$v = e^1 + 2e^2 + 3e^t$ $v \cdot v = (1)^2 + (2)^2 - (3)^2 \neq (1)^2 + (2)^2 + (3)^2$

$v = (e^1 \wedge e^t) + 2(e^2 \wedge e^3)$ $v \cdot v = -(1)^2 + (2)^2 \neq (1)^2 + (2)^2$

If we assume that multiplying a vector by its dual vector results in a pseudoscalar, then its magnitude is the square of the coefficient, which is suitable for the definition of vector norm.

ex) $v = 2e^t,\ v^d = 2(-e^1 \wedge e^2 \wedge e^3),\ vv^d = (2)^2 e^t(-e^1 \wedge e^2 \wedge e^3) = (2)^2 I$

$\|v\| = vv^d, \qquad v^d = v \cdot I$

Extending this simple idea to any arbitrary vector yields a multiplication table.

B. basic multiplication

Basic multiplication where $vv^d = I$.

$e^1(-e^2 \wedge e^3 \wedge e^t) = e^2(-e^3 \wedge e^1 \wedge e^t)$

$= e^3(-e^1 \wedge e^2 \wedge e^t) = e^t(-e^1 \wedge e^2 \wedge e^3)$

$= (e^2 \wedge e^3 \wedge e^t)(-e^1) = (e^3 \wedge e^1 \wedge e^t)(-e^2)$

$= (e^1 \wedge e^2 \wedge e^t)(-e^3) = (e^1 \wedge e^2 \wedge e^3)(-e^t) = I$

$$(\mathbb{e}^1 \wedge \mathbb{e}^2)(-\mathbb{e}^3 \wedge \mathbb{e}^t) = (\mathbb{e}^2 \wedge \mathbb{e}^3)(-\mathbb{e}^1 \wedge \mathbb{e}^t) = (\mathbb{e}^3 \wedge \mathbb{e}^1)(-\mathbb{e}^2 \wedge \mathbb{e}^t)$$

$$= (\mathbb{e}^1 \wedge \mathbb{e}^t)(\mathbb{e}^2 \wedge \mathbb{e}^3) = (\mathbb{e}^2 \wedge \mathbb{e}^t)(\mathbb{e}^3 \wedge \mathbb{e}^1) = (\mathbb{e}^3 \wedge \mathbb{e}^t)(\mathbb{e}^1 \wedge \mathbb{e}^2) = I$$

The product of 1-basis and 3-basis is commutative, and the product of 2-basis and 2-basis is anticommutative. The commutative and associative laws do not apply to multiplication.

C. $1^d,\ I^d$

$$1I = I1 = I = 1\{1^d\} = I\{I^d\}, \qquad 1^d = I, \qquad I^d = 1$$

D. square of basis

ex1) $vv^d = (I + 2)(1 + 2I) = ((1)^2 + (2)^2)I + 2(1 + I^2)$

To satisfy the definition of vector norm $\|I + 2\| = \{(1)^2 + (2)^2\}I$

$$1^2 + I^2 = 0 \qquad\qquad I^2 = -1$$

ex2) $vv^d = \{\mathbb{e}^1 + 2(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)\}\{(-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2(-\mathbb{e}^1)\} = ((1)^2 + (2)^2)I$

$$\mathbb{e}^1\mathbb{e}^1 + (\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)^2 = 0$$

$$(\mathbb{e}^1)^2 = (\mathbb{e}^2)^2 = (\mathbb{e}^3)^2 = (\mathbb{e}^t)^2 = -1$$

$$(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)^2 = (\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t)^2 = (\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)^2 = (\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)^2 = 1$$

$(\mathbb{e}^1)^2 = 1$ is also possible, but it is -1 to explain gravity. 6.D

ex3) $vv^d = \{(\mathbb{e}^1 \wedge \mathbb{e}^t) + 2(\mathbb{e}^2 \wedge \mathbb{e}^3)\}\{(\mathbb{e}^2 \wedge \mathbb{e}^3) - 2(\mathbb{e}^1 \wedge \mathbb{e}^t)\} = ((1)^2 + (2)^2)I$

$vv^d = \{(\mathbb{e}^2 \wedge \mathbb{e}^3) - 2(\mathbb{e}^1 \wedge \mathbb{e}^t)\}\{-(\mathbb{e}^1 \wedge \mathbb{e}^t) + 2(\mathbb{e}^2 \wedge \mathbb{e}^3)\} = ((1)^2 + (2)^2)I$

$$(\mathbb{e}^1 \wedge \mathbb{e}^t)(\mathbb{e}^1 \wedge \mathbb{e}^t) - (\mathbb{e}^2 \wedge \mathbb{e}^3)(\mathbb{e}^2 \wedge \mathbb{e}^3) = 0$$

$$(\mathbb{e}^1 \wedge \mathbb{e}^t)^2 = (\mathbb{e}^2 \wedge \mathbb{e}^3)^2 = 1$$

$$(\mathbb{e}^1 \wedge \mathbb{e}^t)^2 = (\mathbb{e}^2 \wedge \mathbb{e}^t)^2 = (\mathbb{e}^3 \wedge \mathbb{e}^t)^2 = (\mathbb{e}^1 \wedge \mathbb{e}^2)^2 = (\mathbb{e}^2 \wedge \mathbb{e}^3)^2 = (\mathbb{e}^3 \wedge \mathbb{e}^1)^2 = 1$$

This value can also be -1, but to describe the Lorentz transformation and Pauli spin matrices, it must be $+1$. This sign determines spacetime. If it is $+1$, it is hyperbolic geometry, and if it is -1, it is elliptic geometry. 5.2

E. $\mathbb{b}_i I,\ I\mathbb{b}_i$

ex4) $vv^d = (\mathbb{e}^1 + 2)\{(-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2I\} = ((1)^2 + (2)^2)I + 2(-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2\mathbb{e}^1 I$

$vv^d = \{(-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2I\}(\mathbb{e}^1 + 2) = ((1)^2 + (2)^2)I + 2(-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2I\mathbb{e}^1$

To satisfy the definition of vector norm $\|\mathbb{e}^1 + 2\| = \{(1)^2 + (2)^2\}I$

$$\mathbb{e}^1 I = (\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) \qquad I\mathbb{e}^1 = (\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)$$

ex5) $vv^d = \{(-\mathbb{e}^3 \wedge \mathbb{e}^t) + 2\}\{(-\mathbb{e}^1 \wedge \mathbb{e}^2) + 2I\} = ((1)^2 + (2)^2)I$

$vv^d = \{(\mathbb{e}^1 \wedge \mathbb{e}^2) + 2I\}\{(-\mathbb{e}^3 \wedge \mathbb{e}^t) + 2\} = ((1)^2 + (2)^2)I$

$(\mathbb{e}^3 \wedge \mathbb{e}^t)I = -(\mathbb{e}^1 \wedge \mathbb{e}^2) \qquad I(\mathbb{e}^3 \wedge \mathbb{e}^t) = (\mathbb{e}^1 \wedge \mathbb{e}^2)$

ex6) $vv^d = \{(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2\}\{-\mathbb{e}^1 + 2I\} = ((1)^2 + (2)^2)I$

$vv^d = \{-\mathbb{e}^1 + 2I\}\{(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2\} = ((1)^2 + (2)^2)I$

$(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)I = \mathbb{e}^1 \qquad I(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) = \mathbb{e}^1$

F. $\mathbb{b}_i\mathbb{b}_j$

The multiplication of $\mathbb{b}_1, \mathbb{b}_2, \mathbb{b}_3$ cancels out the common basis through the dot product, like the inner product in geometric algebra. In some cases, the signs are reversed. (5.K)

$$(\dots \wedge \mathbb{e}^p \wedge \mathbb{e}^q)(\mathbb{e}^q \wedge \mathbb{e}^r \wedge \dots) = \pm(\mathbb{e}^q \cdot \mathbb{e}^q)\{(\dots \wedge \mathbb{e}^p)(\mathbb{e}^r \wedge \dots)\}, \qquad p, q, r, s = 1,2,3,t$$

G. $\mathbb{b}_1\mathbb{b}_1,\ \mathbb{b}_3\mathbb{b}_3$

$\mathbb{e}^p\mathbb{e}^q = (\mathbb{e}^p \wedge \mathbb{e}^q)$

$\mathbb{e}^q\mathbb{e}^p = (\mathbb{e}^q \wedge \mathbb{e}^p) = -(\mathbb{e}^p \wedge \mathbb{e}^q)$ anti-commutative

$(\mathbb{e}^p \wedge \mathbb{e}^q \wedge \mathbb{e}^r)(\mathbb{e}^p \wedge \mathbb{e}^q \wedge \mathbb{e}^s) = -(\mathbb{e}^r \wedge \mathbb{e}^q \wedge \mathbb{e}^p)(\mathbb{e}^p \wedge \mathbb{e}^q \wedge \mathbb{e}^s)$

$= -(\mathbb{e}^p \cdot \mathbb{e}^p)(\mathbb{e}^r \wedge \mathbb{e}^q)(\mathbb{e}^q \wedge \mathbb{e}^s) = -(\mathbb{e}^p \cdot \mathbb{e}^p)(\mathbb{e}^q \cdot \mathbb{e}^q)\mathbb{e}^r\mathbb{e}^s = -(\mathbb{e}^p \cdot \mathbb{e}^p)(\mathbb{e}^q \cdot \mathbb{e}^q)(\mathbb{e}^r \wedge \mathbb{e}^s)$

$(\mathbb{e}^p \wedge \mathbb{e}^q \wedge \mathbb{e}^s)(\mathbb{e}^p \wedge \mathbb{e}^q \wedge \mathbb{e}^r) = -(\mathbb{e}^p \cdot \mathbb{e}^p)(\mathbb{e}^q \cdot \mathbb{e}^q)(\mathbb{e}^s \wedge \mathbb{e}^r)$ anti-commutative

ex7) $vv^d = \{\mathbb{e}^1 + 2(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)\}\{(-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2(-\mathbb{e}^2)\} = ((1)^2 + (2)^2)I$

$\mathbb{e}^1\mathbb{e}^2 + (\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) = 0$

$\mathbb{e}^1\mathbb{e}^2 = (\mathbb{e}^1 \wedge \mathbb{e}^2)$

$(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) = (\mathbb{e}^1 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)(\mathbb{e}^t \wedge \mathbb{e}^3 \wedge \mathbb{e}^2) = (\mathbb{e}^t \cdot \mathbb{e}^t)(\mathbb{e}^1 \wedge \mathbb{e}^3)(\mathbb{e}^3 \wedge \mathbb{e}^2)$

$= -(\mathbb{e}^3 \cdot \mathbb{e}^3)\mathbb{e}^1\mathbb{e}^2 = -(\mathbb{e}^1 \wedge \mathbb{e}^2)$

$vv^d = \{\mathbb{e}^1 + 2(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)\}\{(-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2(-\mathbb{e}^t)\} = ((1)^2 + (2)^2)I$

$\mathbb{e}^1\mathbb{e}^t + (\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) = 0$

$\mathbb{e}^1\mathbb{e}^t = (\mathbb{e}^1 \wedge \mathbb{e}^t)$

$(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) = -(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)(\mathbb{e}^3 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t)$

$= -(\mathbb{e}^3 \cdot \mathbb{e}^3)(\mathbb{e}^2 \cdot \mathbb{e}^2)\mathbb{e}^1\mathbb{e}^t = -(\mathbb{e}^1 \wedge \mathbb{e}^t)$

H. $\mathbb{b}_2\mathbb{b}_2$

$$(\mathbb{e}^p \wedge \mathbb{e}^q)(\mathbb{e}^p \wedge \mathbb{e}^r) = -(\mathbb{e}^q \wedge \mathbb{e}^p)(\mathbb{e}^p \wedge \mathbb{e}^r) = -(\mathbb{e}^p \cdot \mathbb{e}^p)(\mathbb{e}^q \wedge \mathbb{e}^r)$$

$$(\mathbb{e}^p \wedge \mathbb{e}^r)(\mathbb{e}^p \wedge \mathbb{e}^q) = -(\mathbb{e}^p \cdot \mathbb{e}^p)(\mathbb{e}^r \wedge \mathbb{e}^q) \qquad \text{anti-commutative}$$

Pauli spin matrices are nothing but quaternions.

$$\mathbb{s}^1 = (\mathbb{e}^2 \wedge \mathbb{e}^3), \qquad \mathbb{s}^2 = (\mathbb{e}^3 \wedge \mathbb{e}^1), \qquad \mathbb{s}^3 = (\mathbb{e}^1 \wedge \mathbb{e}^2)$$

$$\mathbb{s}^1\mathbb{s}^2 = (\mathbb{e}^2 \wedge \mathbb{e}^3)(\mathbb{e}^3 \wedge \mathbb{e}^1) = (\mathbb{e}^3 \cdot \mathbb{e}^3)\mathbb{e}^2\mathbb{e}^1 = -(\mathbb{e}^1 \wedge \mathbb{e}^2) = -\mathbb{s}^3$$

$$\mathbb{s}^1\mathbb{s}^1 = (\mathbb{e}^2 \wedge \mathbb{e}^3)(\mathbb{e}^2 \wedge \mathbb{e}^3) = 1, \ \mathbb{s}^2\mathbb{s}^2 = \mathbb{s}^3\mathbb{s}^3 = 1 \qquad 5.D$$

$$\mathbb{s}^2\mathbb{s}^3 = -\mathbb{s}^3\mathbb{s}^2 = -\mathbb{s}^1, \qquad \mathbb{s}^3\mathbb{s}^1 = -\mathbb{s}^3\mathbb{s}^1 = -\mathbb{s}^2, \qquad \mathbb{s}^1\mathbb{s}^2 = -\mathbb{s}^2\mathbb{s}^1 = -\mathbb{s}^3$$

$$\mathbb{s}^1\mathbb{s}^2\mathbb{s}^3 = (\mathbb{s}^1\mathbb{s}^2)\mathbb{s}^3 = \mathbb{s}^1(\mathbb{s}^2\mathbb{s}^3) = -1$$

ex8) $vv^d = \{(\mathbb{e}^1 \wedge \mathbb{e}^2) + 2(\mathbb{e}^2 \wedge \mathbb{e}^t)\}\{(-\mathbb{e}^3 \wedge \mathbb{e}^t) + 2(\mathbb{e}^3 \wedge \mathbb{e}^1)\} = ((1)^2 + (2)^2)I$

$$(\mathbb{e}^1 \wedge \mathbb{e}^2)(\mathbb{e}^3 \wedge \mathbb{e}^1) - (\mathbb{e}^2 \wedge \mathbb{e}^t)(\mathbb{e}^3 \wedge \mathbb{e}^t) = 0$$

$$(\mathbb{e}^1 \wedge \mathbb{e}^2)(\mathbb{e}^3 \wedge \mathbb{e}^1) = (\mathbb{e}^2 \wedge \mathbb{e}^1)(\mathbb{e}^1 \wedge \mathbb{e}^3) = (\mathbb{e}^1 \cdot \mathbb{e}^1)\mathbb{e}^2\mathbb{e}^3 = (\mathbb{e}^2 \wedge \mathbb{e}^3)$$

$$(\mathbb{e}^2 \wedge \mathbb{e}^t)(\mathbb{e}^3 \wedge \mathbb{e}^t) = -(\mathbb{e}^2 \wedge \mathbb{e}^t)(\mathbb{e}^t \wedge \mathbb{e}^3) = -(\mathbb{e}^t \cdot \mathbb{e}^t)\mathbb{e}^2\mathbb{e}^3 = (\mathbb{e}^2 \wedge \mathbb{e}^3)$$

I. $\mathbb{b}_1\mathbb{b}_3, \ \mathbb{b}_3\mathbb{b}_1$

$$\mathbb{e}^p(\mathbb{e}^p \wedge \mathbb{e}^q \wedge \mathbb{e}^r) = (\mathbb{e}^p \cdot \mathbb{e}^p)(\mathbb{e}^q \wedge \mathbb{e}^r)$$

$$(\mathbb{e}^p \wedge \mathbb{e}^q \wedge \mathbb{e}^r)\mathbb{e}^p = (\mathbb{e}^q \wedge \mathbb{e}^r \wedge \mathbb{e}^p)\mathbb{e}^p = (\mathbb{e}^p \cdot \mathbb{e}^p)(\mathbb{e}^q \wedge \mathbb{e}^r) \qquad \text{commutative}$$

ex9) $vv^d = (\mathbb{e}^1 + 2\mathbb{e}^2)\{(-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2(-\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)\} = ((1)^2 + (2)^2)I$

$$\mathbb{e}^1(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) + \mathbb{e}^2(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) = 0$$

$$\mathbb{e}^1(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) = \mathbb{e}^1(\mathbb{e}^1 \wedge \mathbb{e}^t \wedge \mathbb{e}^3) = (\mathbb{e}^1 \cdot \mathbb{e}^1)(\mathbb{e}^t \wedge \mathbb{e}^3) = -(\mathbb{e}^3 \wedge \mathbb{e}^t)$$

$$\mathbb{e}^2(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) = (\mathbb{e}^2 \cdot \mathbb{e}^2)(\mathbb{e}^3 \wedge \mathbb{e}^t) = (\mathbb{e}^3 \wedge \mathbb{e}^t)$$

$$vv^d = \{(-\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) + 2(-\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)\}(\mathbb{e}^t + 2\mathbb{e}^2) = ((1)^2 + (2)^2)I$$

$$(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)\mathbb{e}^2 + (\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)\mathbb{e}^t = 0$$

$$(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)\mathbb{e}^2 = (\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^2)\mathbb{e}^2 = (\mathbb{e}^2 \cdot \mathbb{e}^2)(\mathbb{e}^3 \wedge \mathbb{e}^1) = (\mathbb{e}^3 \wedge \mathbb{e}^1)$$

$$(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)\mathbb{e}^t = (\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)\mathbb{e}^t = (\mathbb{e}^t \cdot \mathbb{e}^t)(\mathbb{e}^3 \wedge \mathbb{e}^1) = -(\mathbb{e}^3 \wedge \mathbb{e}^1)$$

J. $\mathbb{b}_1\mathbb{b}_2, \ \mathbb{b}_2\mathbb{b}_3$

$$\mathbb{e}^p(\mathbb{e}^p \wedge \mathbb{e}^q) = (\mathbb{e}^p \cdot \mathbb{e}^p)\mathbb{e}^q$$

$(\mathbb{e}^p \wedge \mathbb{e}^q)\mathbb{e}^p = -(\mathbb{e}^q \wedge \mathbb{e}^p)\mathbb{e}^p = -\mathbb{e}^q$ anti-commutative

$(\mathbb{e}^p \wedge \mathbb{e}^q)(\mathbb{e}^p \wedge \mathbb{e}^q \wedge \mathbb{e}^r) = -(\mathbb{e}^q \wedge \mathbb{e}^p)(\mathbb{e}^p \wedge \mathbb{e}^q \wedge \mathbb{e}^r) = -(\mathbb{e}^p \cdot \mathbb{e}^p)(\mathbb{e}^q \cdot \mathbb{e}^q)\mathbb{e}^r = -\mathbb{e}^q$

$(\mathbb{e}^p \wedge \mathbb{e}^q \wedge \mathbb{e}^r)(\mathbb{e}^p \wedge \mathbb{e}^q) = -(\mathbb{e}^r \wedge \mathbb{e}^q \wedge \mathbb{e}^p)(\mathbb{e}^p \wedge \mathbb{e}^q) = -(\mathbb{e}^p \cdot \mathbb{e}^p)(\mathbb{e}^q \cdot \mathbb{e}^q)\mathbb{e}^r = -\mathbb{e}^q$

ex10) $vv^d = \{\mathbb{e}^2 + 2(\mathbb{e}^3 \wedge \mathbb{e}^t)\}\{(-\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) + 2(\mathbb{e}^1 \wedge \mathbb{e}^2)\} = ((1)^2 + (2)^2)I$

$vv^d = \{(-\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) + 2(\mathbb{e}^1 \wedge \mathbb{e}^2)\}\{\mathbb{e}^2 + 2(-\mathbb{e}^3 \wedge \mathbb{e}^t)\} = ((1)^2 + (2)^2)I$

$\mathbb{e}^2(\mathbb{e}^1 \wedge \mathbb{e}^2) - (\mathbb{e}^3 \wedge \mathbb{e}^t)(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) = 0$

$(\mathbb{e}^1 \wedge \mathbb{e}^2)\mathbb{e}^2 + (\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)(\mathbb{e}^3 \wedge \mathbb{e}^t) = 0$

$\mathbb{e}^2(\mathbb{e}^1 \wedge \mathbb{e}^2) = -\mathbb{e}^2(\mathbb{e}^2 \wedge \mathbb{e}^1) = -(\mathbb{e}^2 \cdot \mathbb{e}^2)\mathbb{e}^1 = -\mathbb{e}^1$

$(\mathbb{e}^1 \wedge \mathbb{e}^2)\mathbb{e}^2 = (\mathbb{e}^2 \cdot \mathbb{e}^2)\mathbb{e}^1 = \mathbb{e}^1$

$(\mathbb{e}^3 \wedge \mathbb{e}^t)(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) = (\mathbb{e}^3 \wedge \mathbb{e}^t)(\mathbb{e}^t \wedge \mathbb{e}^3 \wedge \mathbb{e}^1) = (\mathbb{e}^t \cdot \mathbb{e}^t)(\mathbb{e}^3 \cdot \mathbb{e}^3)\mathbb{e}^1 = -\mathbb{e}^1$

$(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)(\mathbb{e}^3 \wedge \mathbb{e}^t) = (\mathbb{e}^1 \wedge \mathbb{e}^t \wedge \mathbb{e}^3)(\mathbb{e}^3 \wedge \mathbb{e}^t) = (\mathbb{e}^3 \cdot \mathbb{e}^3)(\mathbb{e}^t \cdot \mathbb{e}^t)\mathbb{e}^1 = -\mathbb{e}^1$

K. $\mathbb{b}_1\mathbb{b}_2,\ \mathbb{b}_2\mathbb{b}_3$

$\mathbb{e}^p(\mathbb{e}^q \wedge \mathbb{e}^r) = (\mathbb{e}^p \wedge \mathbb{e}^q \wedge \mathbb{e}^r)$

$(\mathbb{e}^q \wedge \mathbb{e}^r)\mathbb{e}^p = (\mathbb{e}^q \wedge \mathbb{e}^r \wedge \mathbb{e}^p) = (\mathbb{e}^p \wedge \mathbb{e}^q \wedge \mathbb{e}^r)$ commutative

In particular, the following multiplication changes the sign while canceling out the common basis. anti-commutative

$(\mathbb{e}^p \wedge \mathbb{e}^q)(\mathbb{e}^p \wedge \mathbb{e}^r \wedge \mathbb{e}^s) = -(\mathbb{e}^q \wedge \mathbb{e}^p)(\mathbb{e}^p \wedge \mathbb{e}^r \wedge \mathbb{e}^s) = (\mathbb{e}^p \cdot \mathbb{e}^p)\mathbb{e}^q(\mathbb{e}^r \wedge \mathbb{e}^s)$

$(\mathbb{e}^p \wedge \mathbb{e}^r \wedge \mathbb{e}^s)(\mathbb{e}^p \wedge \mathbb{e}^q) = (\mathbb{e}^r \wedge \mathbb{e}^s \wedge \mathbb{e}^p)(\mathbb{e}^p \wedge \mathbb{e}^q) = -(\mathbb{e}^p \cdot \mathbb{e}^p)(\mathbb{e}^r \wedge \mathbb{e}^s)\mathbb{e}^q$

ex11) $vv^d = \{\mathbb{e}^1 + 2(\mathbb{e}^1 \wedge \mathbb{e}^t)\}\{(-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2(\mathbb{e}^2 \wedge \mathbb{e}^3)\} = ((1)^2 + (2)^2)I$

$vv^d = \{(-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2(\mathbb{e}^2 \wedge \mathbb{e}^3)\}\{\mathbb{e}^1 + 2(-\mathbb{e}^1 \wedge \mathbb{e}^t)\} = ((1)^2 + (2)^2)I$

$\mathbb{e}^1(\mathbb{e}^2 \wedge \mathbb{e}^3) - (\mathbb{e}^1 \wedge \mathbb{e}^t)(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) = 0$

$(\mathbb{e}^2 \wedge \mathbb{e}^3)\mathbb{e}^1 + (\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)(\mathbb{e}^1 \wedge \mathbb{e}^t) = 0$

$\mathbb{e}^1(\mathbb{e}^2 \wedge \mathbb{e}^3) = (\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)$

$(\mathbb{e}^2 \wedge \mathbb{e}^3)\mathbb{e}^1 = (\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^1) = (\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)$

$(\mathbb{e}^1 \wedge \mathbb{e}^t)(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) = (\mathbb{e}^1 \wedge \mathbb{e}^t)(\mathbb{e}^t \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) = -(\mathbb{e}^t \cdot \mathbb{e}^t)\mathbb{e}^1(\mathbb{e}^2 \wedge \mathbb{e}^3) = \mathbb{e}^1(\mathbb{e}^2 \wedge \mathbb{e}^3)$

$(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)(\mathbb{e}^1 \wedge \mathbb{e}^t) = -(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)(\mathbb{e}^t \wedge \mathbb{e}^1) = (\mathbb{e}^t \cdot \mathbb{e}^t)(\mathbb{e}^2 \wedge \mathbb{e}^3)\mathbb{e}^1$

$= -(\mathbb{e}^2 \wedge \mathbb{e}^3)\mathbb{e}^1$

5.1 magnitude of vector norm

The squares of the basis in a dimension are all $+1$ or -1. It depends on the dimension. . 5.D. And the multiplication of $\mathbb{b}_1\mathbb{b}_1$, $\mathbb{b}_2\mathbb{b}_2$ and $\mathbb{b}_3\mathbb{b}_3$ is all anticommutative. 5.G, 5.H. Therefore, the magnitude of a vector norm formed by a basis in a dimension can be expressed as the square of that vector.

ex) $v = \mathbb{e}^1 + 2\mathbb{e}^2 + 3\mathbb{e}^t$

$v^2 = (\mathbb{e}^1 + 2\mathbb{e}^2 + 3\mathbb{e}^t)(\mathbb{e}^1 + 2\mathbb{e}^2 + 3\mathbb{e}^t)$

$= (\mathbb{e}^1)^2 + (2\mathbb{e}^2)^2 + (3\mathbb{e}^t)^2 + 2(\mathbb{e}^1\mathbb{e}^2 + \mathbb{e}^2\mathbb{e}^1) + 6(\mathbb{e}^2\mathbb{e}^t + \mathbb{e}^t\mathbb{e}^2) + 3(\mathbb{e}^1\mathbb{e}^2 + \mathbb{e}^2\mathbb{e}^1)$

$= -\{(1)^2 + (2)^2 + (3)^2\}$ The sign may be different depending on the dimension.

Dual vectors also have the same coefficients, only with different basis. Therefore, the size of a vector norm can be expressed as the square of the dual vector.

$v^d = v \cdot I = 1(-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2(-\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) + 3(-\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)$

$\{v^d\}^2 = \{(1)^2 + (2)^2 + (3)^2\}$

5.2 rotor

Rotor R is a term used in geometric algebra.

5.J

$$\mathbb{e}^1(\mathbb{e}^1 \wedge \mathbb{e}^2) = \mathbb{e}^2, \qquad \mathbb{e}^2(\mathbb{e}^1 \wedge \mathbb{e}^2) = -\mathbb{e}^1$$

$$\mathbb{e}^1(\mathbb{e}^1 \wedge \mathbb{e}^t) = \mathbb{e}^t, \qquad \mathbb{e}^t(\mathbb{e}^1 \wedge \mathbb{e}^t) = \mathbb{e}^1$$

A method of rotating the basis $\mathbb{e}^1, \mathbb{e}^2$ on the plane $(\mathbb{e}^1 \wedge \mathbb{e}^2)$ by $\Delta \ll 1$.

$$\mathbb{e}^{1\prime} = \mathbb{e}^1 R = \mathbb{e}^1\{1 + \Delta(\mathbb{e}^1 \wedge \mathbb{e}^2)\} = \mathbb{e}^1 + \Delta\mathbb{e}^2, \qquad R = \{1 + \Delta(\mathbb{e}^1 \wedge \mathbb{e}^2)\}$$

$$\mathbb{e}^{2\prime} = \mathbb{e}^2 R = \mathbb{e}^2\{1 + \Delta(\mathbb{e}^1 \wedge \mathbb{e}^2)\} = \mathbb{e}^2 - \Delta\mathbb{e}^1$$

If you make a series of small rotations, you get a rotor.

$$R = \{1 + \Delta_1(\mathbb{e}^1 \wedge \mathbb{e}^2)\}\{1 + \Delta_2(\mathbb{e}^1 \wedge \mathbb{e}^2)\} \ldots$$

$$= \lim_{\Delta \to 0}\{1 + \Delta(\mathbb{e}^1 \wedge \mathbb{e}^2)\}^{\frac{\theta}{\Delta}} = e^{(\mathbb{e}^1 \wedge \mathbb{e}^2)\theta}$$

$$= 1 + \theta(\mathbb{e}^1 \wedge \mathbb{e}^2) + \frac{1}{2!}\theta^2(\mathbb{e}^1 \wedge \mathbb{e}^2)^2 + \frac{1}{3!}\theta^3(\mathbb{e}^1 \wedge \mathbb{e}^2)^3 + \cdots$$

$$= \begin{cases} cosh\,\theta + sinh\,\theta\,(\mathbb{e}^1 \wedge \mathbb{e}^2) & ((\mathbb{e}^1 \wedge \mathbb{e}^2)^2 = +1) \\ cos\,\theta + sin\,\theta\,(\mathbb{e}^1 \wedge \mathbb{e}^2) & ((\mathbb{e}^1 \wedge \mathbb{e}^2)^2 = -1) \end{cases}$$

Depending on the sign of $(\mathbb{e}^1 \wedge \mathbb{e}^2)^2$, it is divided into hyperbolic geometry and elliptic geometry.

If $R = e^{(\mathbb{e}^1 \wedge \mathbb{e}^t)\theta}$ and $\tanh\theta = \beta = v/c$ then it is a Lorentz transformation.

$$\mathbb{e}^{1\prime} = \mathbb{e}^1 R = \mathbb{e}^1 e^{(\mathbb{e}^1 \wedge \mathbb{e}^t)\theta} = \mathbb{e}^1\{cosh\,\theta + sinh\,\theta\,(\mathbb{e}^1 \wedge \mathbb{e}^t)\} = \mathbb{e}^1\,cosh\,\theta + \mathbb{e}^t\,sinh\,\theta$$

$$\mathbb{e}^{t\prime} = \mathbb{e}^t R = \mathbb{e}^t e^{(\mathbb{e}^1 \wedge \mathbb{e}^t)\theta} = \mathbb{e}^t\{cosh\,\theta + sinh\,\theta\,(\mathbb{e}^1 \wedge \mathbb{e}^t)\} = \mathbb{e}^t\,cosh\,\theta + \mathbb{e}^1\,sinh\,\theta$$

Note that this is a transformation of the basis, not the coordinates.

$$x'\mathbb{e}^{1\prime} + t'\mathbb{e}^{t\prime} = x\mathbb{e}^1 + t\mathbb{e}^t$$

6 energy density and force

The conventional energy density $eng(A)$ in electromagnetism is given below.

$$eng(A) = \frac{1}{2}(B^2 + E^2) = \frac{1}{2}\{(B_i)^2 + (E_i)^2\}, \qquad i = 1,2,3$$

In this book, it is nothing but the vector norm of $curl(A) = B + E$. Therefore, the energy density is pseudoscalar.

$$eng(A) = \frac{1}{2}\|curl(A)\| = \frac{1}{2}curl(A)curl^d(A)$$

Force is a phenomenon that occurs when energy density is not continuous. 3.E

$$F = -div\{eng(A)\} = -curl\{eng^d(A)\}$$

$eng^d(A)$ is the magnitude of $eng(A)$. It is the norm magnitude of $curl(A)$. This is also the norm magnitude of $curl^d(A)$. 5.1. The sign may change depending on the dimension.

$$eng^d(A) = \frac{1}{2}\{curl(A)\}^2 \quad or \quad \frac{1}{2}\{curl^d(A)\}^2$$

The following assumptions about vector fields X and Y are convenient: again, vectors are numbers.

$$curl(XY) = \{curl(X)\}Y + X\{curl(Y)\}$$

The second case is $eng^d(A)$, which can define the force. The first is because $curl^2(A) = 0$.

$$F = \frac{1}{2}curl\left(\{curl^d(A)\}^2\right) = \frac{1}{2}\left\{curl\{curl^d(A)\}curl^d(A) + curl^d(A)curl\{curl^d(A)\}\right\}$$
$$= \frac{1}{2}\{Lap(A)curl^d(A) + curl^d(A)Lap(A)\} = \frac{1}{2}\{Jcurl^d(A) + curl^d(A)J\}$$

Note that among the terms of $Jcurl^d(A) + curl^d(A)J$, only the 1-vector term is a force. This is because the force is the $curl$ of the scalar.

Again, the force arises where the energy density is not continuous, where $Lap(A) = J \neq 0$, that is, where $curl(A)$ is not continuous, and where the particle is present.

Let's start with electromagnetic force, and then extend it to vector fields in other dimensions.

A. electromagnetic force

In 4.1, $curl^d(A)$ is a 2-vector and J is a 3-vector. And in 5.J , the multiplication of the two vector fields is commutative.

$$eng^d(A) = \frac{1}{2}\{curl^d(A)\}^2, \qquad F = -curl\{eng^d(A)\} = -Jcurl^d(A)$$

$$J = -J_t(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) - J_1(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) - J_2(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) - J_3(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t)$$

$$curl^d(A) = E^d + B^d = E_1(\mathbb{e}^2 \wedge \mathbb{e}^3) + E_2(\mathbb{e}^3 \wedge \mathbb{e}^1) + E_3(\mathbb{e}^1 \wedge \mathbb{e}^2)$$
$$-B_1(\mathbb{e}^1 \wedge \mathbb{e}^t) - B_2(\mathbb{e}^2 \wedge \mathbb{e}^t) - B_3(\mathbb{e}^3 \wedge \mathbb{e}^t)$$

$$F_1\mathbb{e}^1 = J_t(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)E_1(\mathbb{e}^2 \wedge \mathbb{e}^3)$$
$$-J_3(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t)B_2(\mathbb{e}^2 \wedge \mathbb{e}^t)$$
$$-J_2(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)B_3(\mathbb{e}^3 \wedge \mathbb{e}^t) = (-J_tE_1 + J_2B_3 - J_3B_2)\mathbb{e}^1 \qquad 5.K$$
$$F_t\mathbb{e}^t = J_1(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)E_1(\mathbb{e}^2 \wedge \mathbb{e}^3)$$
$$J_2(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)E_2(\mathbb{e}^3 \wedge \mathbb{e}^1)$$
$$J_3(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t)E_3(\mathbb{e}^1 \wedge \mathbb{e}^2) = (-J_1E_1 - J_2E_2 - J_3E_3)\mathbb{e}^t$$

Conventional expression. Here, $F_t\mathbb{e}^t$ is the electromagnetic energy lost per unit time.

$$F = -J_tE + J \times B = \rho E + J \times B, \qquad F_t = -J \cdot E$$

B. weak force

In 4.2, $curl^d(\mathcal{B} + \mathcal{E})$ is a 1-vector and $\mathcal{J}$ is a 2-vector. And this multiplication is anti-commutative. The force cancels out.

The weak force is not a force itself that is weak, but rather forces that cancel each other out. This is a geometric property. And since $F_t\mathbb{e}^t$ is also 0, there is no energy lost to friction or other causes.

The fact that forces are counterbalanced doesn't mean there's no force. It's probably like pushing or pulling the two sides of a spring in opposite directions. A spring is essential for weak force.

Still, let's combine the pushing and pulling forces on both sides of the spring.

$$eng^d(A) = -\frac{1}{2}\{curl^d(\mathcal{B} + \mathcal{E})\}^2, \qquad F = \mathcal{J}curl^d(\mathcal{B} + \mathcal{E})$$

$curl^d(\mathcal{B} + \mathcal{E})$ is a 1-vector, so it has different signs.

$$\mathcal{J} = \mathcal{K} + \mathcal{L}$$

$$\mathcal{K} = -\mathcal{K}_1(\mathbb{e}^2 \wedge \mathbb{e}^3) - \mathcal{K}_2(\mathbb{e}^3 \wedge \mathbb{e}^1) - \mathcal{K}_3(\mathbb{e}^1 \wedge \mathbb{e}^2)$$

$$\mathcal{L} = \mathcal{L}_1(\mathbb{e}^1 \wedge \mathbb{e}^t) + \mathcal{L}_2(\mathbb{e}^2 \wedge \mathbb{e}^t) + \mathcal{L}_3(\mathbb{e}^3 \wedge \mathbb{e}^t)$$

$$curl^d(\mathcal{B} + \mathcal{E}) = -\mathcal{A}_1\mathbb{e}^1 - \mathcal{A}_2\mathbb{e}^2 - \mathcal{A}_3\mathbb{e}^3 - \mathcal{A}_t\mathbb{e}^t$$

$$F_1\mathbb{e}^1 = \mathcal{L}_1(\mathbb{e}^1 \wedge \mathbb{e}^t)\mathcal{A}_t\mathbb{e}^t - \mathcal{K}_2(\mathbb{e}^3 \wedge \mathbb{e}^1)\mathcal{A}_3\mathbb{e}^3 - \mathcal{K}_3(\mathbb{e}^1 \wedge \mathbb{e}^2)\mathcal{A}_2\mathbb{e}^2$$

$$= (-\mathcal{L}_1\mathcal{A}_t + \mathcal{K}_2\mathcal{A}_3 - \mathcal{K}_3\mathcal{A}_2)\mathbb{e}^1 \qquad 5.K$$

Conventional expression

$$F = -\mathcal{L}\mathcal{A}_t + \mathcal{K} \times \mathcal{A}$$

C. strong force

In 4.3, $curl^d(\mathbb{A})$ is a scalar and $\mathbb{J}$ is a 1-vector. The product of the two vector fields is commutative.

$$eng^d(\mathbb{A}) = \frac{1}{2}\{curl^d(\mathbb{A})\}^2, \qquad F = -\mathbb{J}curl^d(\mathbb{A})$$

$$\mathbb{J} = \mathbb{J}_1\mathbb{e}^1 + \mathbb{J}_2\mathbb{e}^2 + \mathbb{J}_3\mathbb{e}^3 + \mathbb{J}_t\mathbb{e}^t, \qquad curl^d(\mathbb{A}) = \mathbb{V}$$

$$F_1\mathbb{e}^1 = -\mathbb{J}_1\mathbb{e}^1\mathbb{V} = -\mathbb{V}\mathbb{J}_1\mathbb{e}^1, \qquad F_t\mathbb{e}^t = -\mathbb{J}_t\mathbb{e}^t\mathbb{V} = -\mathbb{V}\mathbb{J}_t\mathbb{e}^t$$

Conventional expression

$$F = -\mathbb{V}\mathbb{J}, \qquad F_t = -\mathbb{V}\mathbb{J}_t$$

In mathematical terms, the color density itself is a force, and the Higgs density appears to be the energy lost per time.

D. gravitational force

In 4.4, $curl^d(\mathcal{v})$ is a 3-vector and $\mathcal{j}$ is a pseudoscalar. The product of the two vector fields is commutative.

$$eng^d(\mathcal{v}) = \frac{1}{2}\{curl^d(\mathcal{v})\}^2, \qquad F = -\mathcal{j}curl^d(\mathcal{v})$$

$$\mathcal{j} = -\mathcal{m}I$$

$$curl^d(\mathcal{v}) = -a_t(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) - a_1(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) - a_2(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) - a_3(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t)$$

$$F_1\mathbb{e}^1 = -\mathcal{m}Ia_1(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) = -\mathcal{m}a_1\mathbb{e}^1 = \mathcal{m}(-\partial_1\mathcal{v})\mathbb{e}^1 \qquad 5.E$$

$$F_t\mathbb{e}^t = -\mathcal{m}Ia_t(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) = -\mathcal{m}a_t\mathbb{e}^t = \mathcal{m}(-\partial_t\mathcal{v})\mathbb{e}^t$$

Conventional expression

$$F = -ma = -m(\nabla v), \qquad F_t = -m(\partial_t v)$$

In 4.4. m is a pseudoscalar dimension. It is also a dimension of energy density. In the well-known kinetic energy $mv^2/2$ and potential energy mgh, m is the only energy density. v and gh can be interpreted simply as the ratio of time and length.

The same goes for momentum mv. It is not the law of conservation of momentum, but another law of conservation of energy.

I hope you find the error. And if not, believe.